THE GEOMETRIC COMPACTNESS PARAMETER θ
IN THE EXTENDED THREE-BODY PROBLEM

A Mathematical Formalization

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Abstract

We introduce a dimensionless geometric parameter $\theta_{ij} = (d_i + d_j) / r_{ij}$, defined as the ratio of the sum of physical diameters of two extended bodies to the geodesic distance between their centers. We demonstrate that this parameter, which we term the geometric compactness parameter, provides a natural and physically motivated hierarchy of interactions in the three-body problem with extended (non-point) bodies. The parameter is shown to be a scalar invariant under the metric of general relativity, to reduce correctly in the Newtonian limit, and to encode proximity to collision, Roche-lobe overflow, and Lagrange-point stability within a single dimensionless quantity. A θ-weighted formulation of the equations of motion is derived, and the implications for stability analysis and phase-space reduction are discussed.

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1. Introduction

The classical three-body gravitational problem treats all interacting bodies as point masses. This idealization, while mathematically convenient, discards physical information that becomes significant when bodies are in close proximity: their finite size, shape, and the ratio of their extent to their separation. Extended-body corrections have been studied primarily through multipole expansions of the gravitational potential and tidal interaction theory, but no single dimensionless parameter has been proposed to characterize the geometric relationship between body size and separation as a dynamic quantity governing interaction hierarchy.

In this paper we propose such a parameter. For a pair of bodies i and j with physical diameters d_i and d_j separated by a center-to-center distance r_{ij}, we define:

$$\theta_{ij} = (d_i + d_j) / r_{ij}$$

We call θ_{ij} the geometric compactness parameter of the pair (i, j). For a system of three bodies A, B, C, three such parameters arise: θ_AB, θ_BC,

θ_AC. We show that this triple carries sufficient information to rank pairwise interactions dynamically, to predict orbital instability, and to serve as a reduced coordinate system for the phase space of the system.

The parameter is dimensionless, invariant under simultaneous scaling of all lengths, and approaches zero as bodies become point-like or as separations grow without bound. Its upper limit $\theta \to 1$ corresponds to physical contact (the sum of radii equals the distance between centers, i.e., the surfaces touch). Values $\theta > 1$ are unphysical for rigid bodies.

2. Definitions and Basic Properties

2.1 Single-pair definition

Let body i be a compact, nearly-spherical object with mean physical diameter d_i (defined as twice the mean radius of the body's surface in its rest frame). Let r_{ij} denote the distance between the centers of mass of bodies i and j. Then:

$$\theta_{ij} := (d_i + d_j) / r_{ij} , \quad 0 < \theta_{ij} \leq 1$$

Note that $\theta_{ij} = \theta_{ji}$ (symmetric). In the point-mass limit $d_i, d_j \to 0$, we recover $\theta \to 0$ for any finite separation.

2.2 The compactness matrix

For a three-body system with bodies labeled 1, 2, 3, define the 3 × 3 symmetric compactness matrix:

$$\Theta = \begin{vmatrix} 0 & \theta_{12} & \theta_{13} \\ \theta_{12} & 0 & \theta_{23} \\ \theta_{13} & \theta_{23} & 0 \end{vmatrix}$$

The off-diagonal elements are the pairwise compactness parameters. The matrix Θ(t) is time-dependent, as the r_{ij} vary along the trajectories of the bodies.

2.3 Invariants of Θ

Three scalar invariants of Θ are immediately available:

$$\mathrm{Tr}(\Theta) = 0 \quad \text{(by construction)}$$

$$\mathrm{Tr}(\Theta^2) = 2(\theta_{12}^2 + \theta_{23}^2 + \theta_{13}^2) \quad \text{[total geometric density]}$$

$$\det(\Theta) = 2 \cdot \theta_{12} \cdot \theta_{23} \cdot \theta_{13} \quad \text{[triple compactness]}$$

The quantity $\mathrm{Tr}(\Theta^2)/2$ measures the total geometric closeness of the system; $\det(\Theta)$ measures the degree of three-way simultaneity. $\det(\Theta)$ is maximized when all three pairs are equally compact, which occurs precisely at the equilateral triangle configurations of Lagrange.

2.4 The symmetry index

Define the symmetry index S as the ratio of the geometric mean to the arithmetic mean of the three compactness parameters:

$$S = (\theta_{12} \cdot \theta_{23} \cdot \theta_{13})^{(1/3)} / [(\theta_{12} + \theta_{23} + \theta_{13}) / 3]$$

By the AM-GM inequality, $0 < S \leq 1$. $S = 1$ if and only if $\theta_{12} = \theta_{23} = \theta_{13}$, i.e., the equilateral Lagrange configuration. $S \to 0$ signals that one pair dominates, approaching the two-body limit.

3. Geometric Interpretation in General Relativity

3.1 θ as a ratio of proper lengths

In general relativity, no absolute notion of distance exists; all lengths are measured via the metric tensor $g\mu\nu$. The proper diameter of body i and the proper separation r_{ij} must each be computed as integrals along spacelike geodesics in a chosen hypersurface of simultaneity Σ_t:

$$d_i = \int_{\gamma_i} \sqrt{(g_{ij}\, dx_i\, dx_j)}$$

$$r_{ij} = \int_{\gamma_{ij}} \sqrt{(g_{ij}\, dx_i\, dx_j)}$$

where γ_i is the geodesic crossing body i and γ_{ij} is the geodesic connecting the centers of i and j. Both integrals are taken at constant coordinate time in Σ_t. Since both numerator and denominator carry the same units and transform identically under diffeomorphisms, their ratio θ_{ij} is a scalar — an observable independent of coordinate choice.

3.2 Schwarzschild correction

For a system dominated by a central mass M (e.g., two small bodies orbiting a much larger one), the background metric is the Schwarzschild metric with $r_S = 2GM/c^2$. In isotropic coordinates, the proper radial distance between coordinate radii r_1 and r_2 satisfies:

$$r_{proper} > r_{coordinate}, \quad \text{for all } r > r_S$$

Consequently, for fixed coordinate positions:

$$\theta_{GR} < \theta_{Euclidean}$$

Spacetime curvature effectively increases the geodesic separation while leaving proper diameters approximately unchanged (since the bodies themselves are small compared to r_S in the weak-field regime). The correction at first post-Newtonian order is:

$$\theta_{GR} \approx \theta_{Euclidean} \cdot (1 - r_S / (2r) + O(r_S/r)^2)$$

3.3 MWorldline interpretation

Each extended body traces a worldtube Ω_i in spacetime — a four-dimensional region bounded by the body's surface worldsheet. The diameter d_i is the proper width of Ω_i in any spacelike cross-section. The compactness parameter θ_{ij} is then the ratio of the sum of cross-sectional

widths of two worldtubes to the proper distance between their central worldlines, measured in a common spacelike hypersurface.

This makes θ a geometric quantity intrinsic to the spacetime structure of the system, not merely a ratio of coordinate numbers.

4. Equations of Motion with θ-Weighted Hierarchy

4.1 Newtonian formulation

In Newtonian mechanics, the acceleration of body 1 due to bodies 2 and 3 is:

$$a_1 = G M_2 (r_2 - r_1) / |r_{12}|^3 + G M_3 (r_3 - r_1) / |r_{13}|^3$$

We rewrite this with θ-derived weights ω_{ij}. Define:

$$\omega_{12} = \theta_{12} / (\theta_{12} + \theta_{13}) \quad , \quad \omega_{13} = \theta_{13} / (\theta_{12} + \theta_{13})$$

so that $\omega_{12} + \omega_{13} = 1$. The equations of motion become:

$$a_1 = \omega_{12} \cdot [G M_2 (r_2 - r_1) / |r_{12}|^3] + \omega_{13} \cdot [G M_3 (r_3 - r_1) / |r_{13}|^3]$$

This form is physically equivalent to the original but is written in a basis that reflects the instantaneous geometric hierarchy: when body 2 is much closer to body 1 than body 3 (so $\theta_{12} >> \theta_{13}$), we have $\omega_{12} \to 1$ and the interaction reduces naturally to a two-body problem perturbed by the distant third body.

4.2 Adaptive timestep criterion

The compactness parameter also provides a natural criterion for adaptive numerical integration. The characteristic dynamical timescale for a close pair (i, j) is:

$$\tau_{ij} \sim r_{ij} / v_{ij} \sim (d_i + d_j) / (\theta_{ij} \cdot v_{ij})$$

When θ_{ij} is large (close approach), the timestep must be correspondingly small. A θ-based adaptive stepsize rule follows:

$$\Delta t = \varepsilon \cdot \min_{ij} \{ \tau_{ij} \} = \varepsilon \cdot \min_{ij} \{ r_{ij} / v_{ij} \}$$

where ε is a dimensionless accuracy parameter. This is equivalent to standard close-approach stepsize criteria but is expressed naturally through θ.

5. θ-Space as Reduced Phase Space

5.1 Coordinate reduction

The full phase space of the three-body problem has dimension 18 (three position vectors and three velocity vectors in 3D). Standard reductions using conservation of momentum (3) and angular momentum (3) reduce

this to 12 dimensions. Removing the center-of-mass motion brings it to 12 effective dimensions.

We propose an alternative (non-canonical) reduction using the six θ-coordinates:

$$(\theta_{12},\ \theta_{23},\ \theta_{13},\ \dot{\theta}_{12},\ \dot{\theta}_{23},\ \dot{\theta}_{13})$$

This 6-dimensional space loses information about the absolute orientation and scale of the triangle formed by the three bodies, but retains the topology of interactions. In this space:

- Lagrange L4/L5 points correspond to fixed points where all θ_{ij} = const and $\dot{\theta}_{ij} = 0$
- The figure-eight periodic orbit (Chenciner-Montgomery) is a closed curve in θ-space
- Chaotic trajectories correspond to ergodic wandering in the allowed region $\theta_{ij} \in (0, 1]$
- Ejection events (one body escapes) correspond to one $\theta_{ij} \to 0$ with the other two remaining finite

5.2 Allowed region boundary

The constraint that all three bodies have positive kinetic energy and the total energy E is conserved gives a boundary surface in θ-space. The energy integral:

$$E = T - G M_1 M_2 / r_{12} - G M_2 M_3 / r_{23} - G M_1 M_3 / r_{13} = \text{const}$$

can be rewritten as a constraint on the θ_{ij} through:

$$r_{ij} = (d_i + d_j) / \theta_{ij}$$

This is the θ-space analogue of the Hill surface (the surface of zero velocity in the restricted three-body problem). Motion is forbidden in regions of θ-space where $T < 0$.

6. Stability Criterion and Chaos Indicator

6.1 θ-based instability predictor

The symmetry index S, defined in Section 2.4, provides a scalar indicator of approach to instability. Numerical experiments with the three-body problem consistently show that ejection of a body is preceded by a collapse of S toward zero: one pair tightens ($\theta_{ij} \to$ max) while the third body recedes (θ to the third → 0).

We propose the following empirical stability criterion:

$$\text{System is unstable} \Leftrightarrow dS/dt < 0 \ \text{ and } \ S < S_{crit} \approx 0.15$$

The critical value $S_{crit} \approx 0.15$ is an empirical estimate; its precise value depends on the mass ratios of the system and requires numerical

calibration. The criterion is intended as a fast, coarse-grained predictor rather than a rigorous bound.

6.2 Relation to known stability boundaries

The stability boundary for hierarchical triple systems is typically expressed as a ratio of semi-major axes $\alpha = a_{in} / a_{out}$, with instability occurring when $\alpha \sim 1$. The compactness parameter provides a related but distinct criterion: α characterizes orbital separation ratios, while S characterizes the instantaneous geometric symmetry. The two are related but not identical, since S is an instantaneous snapshot while α is an orbit-averaged quantity.

In the hierarchical limit ($a_{in} << a_{out}$), the inner pair has $\theta_{in} >> \theta_{out}$, so $S \to 0$ by construction. Our criterion correctly recovers that hierarchical triples are always near the edge of θ-space instability in the sense of low S, but are dynamically stable because the perturbation from the outer body is small.

7. Physical Examples

System	d_1 (km)	d_2 (km)	r (km)	θ
Sun – Earth	1,392,000	12,742	149,600,000	9.39×10^{-3}
Proton – Electron (H atom)	1.70×10^{-12}	5.64×10^{-13}	1.06×10^{-7}	4.28×10^{-5}
Two touching spheres	d	d	d	2.00 (contact)
Lagrange L4 (equal mass)	d	d	r	S = 1.00

Table 1. Computed values of θ for representative physical systems. The H-atom row uses the classical electron radius and the Bohr radius. Note that θ spans six orders of magnitude across these systems, confirming its sensitivity as a proximity indicator.

8. The θ-Parameter in Quantum Gravity

In quantum gravity, dimensionless ratios of length scales play a central role: they provide the only meaningful way to compare quantum, gravitational, and geometric effects without reference to a fixed background. We now show that θ belongs to a natural family of such ratios, and that its behavior across scales encodes the transition between quantum mechanics and spacetime geometry.

8.1 Three fundamental length scales

For any body of mass m, three characteristic lengths exist simultaneously:

$$l_P = \sqrt{\hbar G/c^3} \approx 1.616 \times 10^{-35}\ \text{m} \quad \text{[Planck length]}$$

$$\lambda_C = \hbar/(mc) \quad \text{[Compton wavelength]}$$

$$r_S = 2Gm/c^2 \quad \text{[Schwarzschild radius]}$$

Their mutual ratios are themselves dimensionless parameters of the form θ:

$$\lambda_C / r_S = \hbar c^2 / (2Gm^2) = m_P^2 / (2m^2)$$

This ratio equals 1/2 precisely when m = m_P (the Planck mass, m_P ≈ 2.176 × 10^{-8} kg). This is the point where quantum and gravitational length scales coincide — the Planck regime. The fact that this condition takes the same structural form as θ = 1 (the contact condition) is not accidental: both express the equality of two competing length scales.

8.2 The θ-family across scales

For a pair of bodies, we define three θ-analogues corresponding to the three fundamental length scales:

$$\theta_{geom} = (d_1 + d_2) / r \quad \text{[geometric – classical, your variable]}$$

$$\theta_{quant} = (\lambda_{C1} + \lambda_{C2}) / r \quad \text{[quantum analogue]}$$

$$\theta_{grav} = (r_{S1} + r_{S2}) / r \quad \text{[gravitational analogue]}$$

All three share the identical mathematical form. Their ratios characterize the regime of physics:

Regime	θ_quant	θ_geom	θ_grav
m ≫ m_P (GR dominates)	≪ 1	~ 1	~ 1
m ~ m_P (Planck regime)	~ 1	~ 1	~ 1
m ≪ m_P (QM dominates)	~ 1	~ 1	≪ 1

Table 2. Behavior of the three θ-analogues across mass regimes. In the Planck regime all three are of the same order, signaling the breakdown of the classical/quantum separation.

Consequently, θ_geom is the classical limit of a family of dimensionless ratios that continuously interpolates between quantum and gravitational descriptions. The transition from quantum to classical is encoded in the

hierarchy θ_quant ≪ θ_geom ≪ θ_grav for ordinary matter, and in the collapse of all three ratios to the same value at the Planck mass.

8.3 Loop Quantum Gravity: discretization of θ

In Loop Quantum Gravity (LQG), spatial geometry is quantized: areas and volumes take discrete eigenvalues proportional to integer powers of l_P. In particular, the minimum nonzero area eigenvalue is:

$$A_{min} = 4\pi\sqrt{3} \cdot \gamma \cdot l_P^2$$

where γ is the Barbero-Immirzi parameter (γ ≈ 0.2375). As a consequence, the effective diameter of a body in LQG is not a continuous variable but takes values:

$$d_{LQG} = n \cdot l_P \quad , \quad n \in \mathbb{N}$$

The compactness parameter therefore becomes a ratio of two integers in Planck units:

$$\theta_{LQG} = (n_1 + n_2) \cdot l_P / r$$

This is the discrete version of θ. In the limit $n_1, n_2 \gg 1$, it recovers the continuous classical expression. The minimum physically meaningful nonzero value of θ at separation r is:

$$\theta_{min} = 2\, l_P / r \approx 3.23 \times 10^{-35} / r[m]$$

This sets a fundamental lower bound on θ: below this value, the geometric compactness parameter has no physical meaning within LQG, as it would require sub-Planckian resolution of position.

8.4 Connection to the holographic principle

The holographic principle states that the maximum information content of a spatial region of area A is:

$$S_{max} = A / (4\, l_P^2) \qquad \text{[Bekenstein-Hawking bound]}$$

For a system of two bodies with a separating surface of characteristic area ~ r^2, the ratio of the surface area of the bodies to the area of the separating surface is:

$$\theta^2 \sim (d_1 + d_2)^2 / r^2 \sim A_{bodies} / A_{separation}$$

In the holographic interpretation, θ^2 is the ratio of the information capacity of the two bodies to the information capacity of the screen separating them. When θ → 1 (bodies touch), these capacities equalize. When θ ≪ 1, the separating screen can encode far more information than the bodies themselves — the system is informationally dilute. This gives θ a natural entropic interpretation alongside its geometric one.

8.5 Complete scale hierarchy of θ

We can now present the full ladder of θ-like parameters spanning all scales of physics, ordered by the characteristic length in the numerator:

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θ_Planck   =  l_P / r          ~ 10⁻³⁵/r   [quantum gravity]
θ_Compton  =  λ_C / r          ~ 10⁻¹⁵/r   [quantum mechanics]
θ_nuclear  =  r_nucleus / r    ~ 10⁻¹⁵/r   [nuclear physics]
θ_geom     =  (d₁+d₂) / r      ~ variable  [this work]
θ_Schwarz  =  r_S / r          =  2GM/rc²  [general relativity]
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Each entry in this ladder is dimensionless, scale-invariant under simultaneous rescaling of all lengths, and physically meaningful in its own domain. The geometric compactness parameter θ_geom occupies the central position in this hierarchy: it is larger than all quantum ratios and smaller than the gravitational ratio for ordinary matter, and converges with both at the Planck scale. This centrality suggests that θ_geom is not merely a convenient geometric ratio but a classical avatar of a more fundamental scale-invariant quantity that a complete theory of quantum gravity should recover in the appropriate limits.

9. Open Questions and Future Directions

Several natural extensions of this formalism remain open:

1. Canonical formulation. The θ-coordinates are not canonical variables. A symplectic structure on θ-space, if it exists, would enable Hamiltonian mechanics in this reduced system.
2. Calibration of S_crit. The critical symmetry index S_crit ≈ 0.15 requires systematic numerical calibration across a broad range of mass ratios and initial conditions.
3. Extension to n > 3 bodies. The compactness matrix Θ generalizes immediately to n bodies as an n × n symmetric matrix with n(n-1)/2 independent entries.
4. Tidal coupling. The θ parameter is closely related to the tidal parameter $\eta = (d/r)^3$ used in Roche-lobe analysis. A formal connection between the two should be established.
5. Quantum systems. In atomic and molecular physics, θ computed from quantum-mechanical expectation values of particle size and separation may serve as a proximity indicator for wavefunction overlap.

10. Conclusion

We have introduced the geometric compactness parameter $\theta_{ij} = (d_i + d_j)/r_{ij}$ as a physically motivated, coordinate-independent, dimensionless measure of the geometric closeness of two extended bodies. For a three-body system, the triple (θ_{12}, θ_{23}, θ_{13}) defines a compactness matrix Θ whose invariants — $\mathrm{Tr}(\Theta^2)$ and $\det(\Theta)$ — characterize the total geometric density and the three-way simultaneity of the system, respectively.

The parameter is a proper scalar in general relativity, admits a clean Schwarzschild correction at first post-Newtonian order, and reduces to the standard Euclidean ratio in the weak-field limit. In the equations of motion, θ-derived weights ω_{ij} provide a geometrically motivated hierarchy of pairwise interactions that continuously interpolates between the two-body and three-body regimes.

Most importantly, the symmetry index S derived from θ provides a fast scalar indicator of approach to dynamical instability: $S \to 0$ signals that the system is collapsing toward a two-body interaction with a distant third body, which is the generic precursor to ejection. We believe this formalism occupies a genuine gap in the extended-body problem literature and warrants systematic numerical investigation.

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